Lecture 21

Friday, November 22, 2019 5:30 AM

Cauchy's Thm and Integral formula.

Cauchy's Integral Formula – I. Let f be analytic in $G \subseteq \mathbb{C}$ and $\gamma:[0,1]\to G$ a p-w smooth closed curve s.t. $n(\gamma;z)=0$, $\forall z\in\mathbb{C}\setminus G$. Then, $\forall z\in G\setminus\{\gamma\}$:

$$f(z)\,n(\gamma;z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

$D_0 = \{n(\gamma;z) < \varepsilon\}$ open!

Pf. Note, by assumption, $\mathbb{C} = G \cup D_0$, $D_0 = \{z\in\mathbb{C}\setminus\{\gamma\} : n(\gamma;z)=0\}$. We define on $\mathbb{C}$:

$$g(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(z)-f(\zeta)}{z-\zeta}\,d\zeta, & z\in G \\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z}, & z\in D_0 \end{cases}$$

(see below* for def. $z\in\{\gamma\}$)

Claim. $g(z)$ is entire (analytic in $\mathbb{C}$).

- First, check g is well-defined. Pick $z\in G\cap D_0$:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(\zeta)}{z-\zeta}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z} - f(z)\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta-z}}_{n(\gamma;z)=0 \text{ in } D_0} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z}. \ \checkmark$$

- g is analytic in D_0 by C-Leibniz. ✓

- *To see g analytic in G, we consider $\varphi(z,\zeta) := \dfrac{f(z)-f(\zeta)}{z-\zeta}$ in $G\times G$, where $\varphi(z,z)=f'(z)$. Then, φ is cont. in $G\times G$. We claim $\forall \zeta\in G$, $\psi(z)=\varphi(z,\zeta)$ is analytic. ψ is clearly anal. in $G\setminus\{\zeta\}$. Suffices to check ψ is anal. in $B(\zeta,r)$ where $\overline{B(\zeta,r)}\subseteq G$. This is easy

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\zeta)^n;\quad a_n = \frac{f^{(n)}(\zeta)}{n!} \Rightarrow f(z)-f(\zeta) = \sum_{n=1}^{\infty} a_n (z-\zeta)^n$$

$$f(z)-f(\zeta) = \sum_{n=1}^{\infty} a_n (z-\zeta)^n = (z-\zeta)\,h(z),$$ where h is anal. in $B(a,r)$.

$$= (z-\zeta)\underbrace{\sum_{n=0}^{\infty} a_{n+1}(z-\zeta)^n}_{h(z):\ R.O.C. \geq r} = (z-\zeta)\,h(z), \quad \text{where } h \text{ is anal. in } B(a,r).$$

But $\varphi(z) = \dfrac{f(z)-f(\zeta)}{z-\zeta} = h(z) \Rightarrow$ $\underline{\varphi \text{ anal in } G}$. The $\overset{*}{\underline{\text{precise}}}$ def of g:

$g(z) = \frac{1}{2\pi i}\int_\gamma \varphi(z,\zeta)\,d\zeta$ is analytic in G by $\mathbb{C}$-Leibniz.

<u>Claim 2</u>. g is constant and, in fact, $g \equiv 0$.

- The unbdd comp. D_∞ of $\mathbb{C}\setminus\{\gamma\}$ is contained in D_0. As in that proof, when $|z|\to\infty$ ($z\in D_0$ eventually),

$$|g(z)| \leq \frac{1}{2\pi}\int_\gamma \frac{|f(\zeta)|\,|d\zeta|}{|\zeta - z|} \leq \underbrace{\left(\max_{\zeta\in\{\gamma\}} \frac{1}{|z|-|\zeta|}\right)}_{0 \text{ as } |z|\to\infty} \cdot \underbrace{M}_{\max_{\zeta\in\{\gamma\}}|f(\zeta)|} \cdot \underbrace{\text{length of } \{\gamma\}}_{\int_\gamma |d\zeta|}$$

In particular, $|g(z)| \leq 1$ in $\{|z|\geq R\}$; $|g(z)| \leq \max\left(1, \max_{|z|\leq R} |g(z)|\right)$

$\Rightarrow g$ is constant by Liouville. But $g(z)\to 0$ as $z\to\infty \Rightarrow g\equiv 0$, as claimed.

Thus, if $z \in G\setminus\{\gamma\}$, we get

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(\zeta)}{z-\zeta}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z)\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}}_{n(\gamma;z)}. \quad \blacksquare$$

<u>CIF-II</u>. f anal. in G, $\gamma_1,\dots,\gamma_n$ closed, p-w smooth curves in G s.t. $n(\gamma_1;z)+\dots+n(\gamma_n;z)=0$ in $\mathbb{C}\setminus G$. Then, $\forall z\in G\setminus\bigcup_{k=1}^n\{\gamma_k\}$

$$\left(\sum_{k=1}^n n(\gamma_k;z)\right) f(z) = \sum_{k=1}^n \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

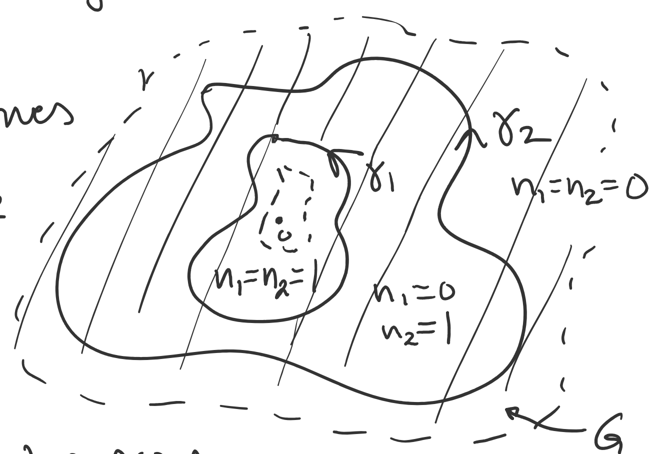
Pf. Just use $\gamma = \gamma_1 + \dots + \gamma_n$ in pf above ☒

Cauchy's Thm – I. Let $f, G, \gamma_1, \dots, \gamma_n$ be as in CIF – II. Then

$$\sum_{k=1}^{n} \int_{\gamma_k} f\,dz = 0$$

Pf. Pick $a \in G \setminus \bigcup_{1}^{n} \{\gamma_k\}$ and consider $g(z) = (z-a) f(z)$ in CIF – II. ☒

Typical Ex. Consider 2 closed curves



Let G be shaded region (w/ hole inside γ_1). Then, $n(-\gamma_1, z) + n(\gamma_2, z) = 0$ in $G \setminus \{\gamma_1\} \cup \{\gamma_2\}$. Thus, if f anal. in G,

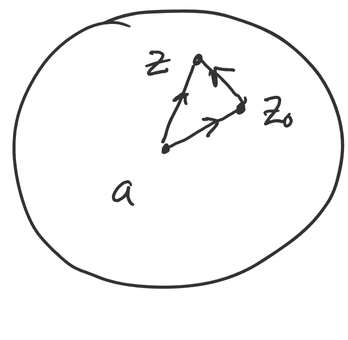
$$\big(n(\gamma_2, z) - n(\gamma_1, z) \big) f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

Converse to CT – I.

Morera's Thm. Let f be cont. in G. If, for every $B(a,r) \subseteq G$ and every triangular path $\Gamma = [a_1, a_2] \cup [a_2, a_3] \cup [a_3, a_1]$ in $B(a,r)$, we have $\int_{\Gamma} f\,dz = 0$, then f is analytic in G.

Pf. Suffices to show f analytic in $B(a,r) \subseteq G$. We shall show $\exists$ anal. fcn F in $B(a,r)$ s.t. $F' = f$. This completes proof.

Set $F(z) = \int_{[a,z]} f\,dz$



$F(z)$ $F(z_0)$

$$\overset{F(z)}{\int_{[a,z]} f\,d\zeta} = \overset{F(z_0)}{\int_{[a,z_0]} f\,d\zeta} + \int_{[z_0,z]} f\,d\zeta \Rightarrow$$

By assumption, the identity above holds.

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} \big(f(\zeta)-f(z_0)\big)\,d\zeta .$$

$\forall \varepsilon > 0\ \exists\, \delta > 0$ s.t. $\max\limits_{\zeta\in[z_0,z]} |f(\zeta)-f(z_0)| < \varepsilon$ when $|z-z_0|<\delta \Rightarrow$

$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| < \varepsilon$ when $|z-z_0|<\delta \Rightarrow F$ $\mathbb{C}$-diff at z_0 and $F'(z_0) = f(z_0)$. Since f is cont, $z_0 \in B(a,r)$ arbitrary, $\Rightarrow F$ is analytic in $B(a,r) \Rightarrow f = F'$ is analytic in $B(a,r)$. $\boxtimes$